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Totally Positive Completable Matrix Patterns and Expansion

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Abstract

Though some special cases are now understood, the characterization of TP-completable patterns is far from complete. Here, a new idea is developed: the expansion of a pattern. It is used to explain some recent results, such as border patterns. The effects of expansion on certain cases of non-completable and completable patterns is examined, as well as an attempt to characterize 3-by-n TP-completable patterns. While many TP-completable patterns remain so under expansion, a counterExample shows that this is not always so. In the process, some new results about TP-completability are given.

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Chapter 1

Introduction

An m-by-n matrix is called *totally positive*, TP (*totally nonnegative*, TN), if all its minors are positive (nonnegative). A matrix is TP_k if each minor of size less than or equal to k is positive. Many characterizations exist that facilitate checking a matrix for total positivity. If all minors based on contiguous row and column index sets are positive, then all minors are positive [3]. Further, it is sufficient if among these, only the "initial" minors [7] are positive. A *partial matrix* is one in which some entries are specified and the remaining unspecified entries are free to be chosen [1]. A *completion* of a partial matrix is a choice of values for the unspecified entries, resulting in a conventional matrix. A matrix completion problem asks which partial matrices have a completion with a given desired property. Here, we examine the totally positive completion problem. Because each submatrix of a TP (TN) matrix must also be TP (TN), it is necessary for a partial matrix to be *partial TP (TN)* in order to have a TP (TN) completion. This means that each minor consisting only of specified entries is positive. We can develop analogous labels for PTP_k as we did with TP_k. The *pattern* of a partial TP matrix will be important. This is just the set of positions of the specified entries. We use $\mathcal{P}$ to denote a pattern, and use "x" to denote a specified position and “?” to denote an unspecified position. For Example,

$$\begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & x \end{bmatrix}$$

is a 3-by-3 pattern. So a pattern may be viewed as a matrix consisting of x's and ?'s.

For a pattern $\mathcal{P}$ to be TP (TN) completable, we mean that every partial TP (TN) matrix with the pattern $\mathcal{P}$ has a TP (TN) completion. This happens for some patterns and not for others. One major problem is to identify the TP-completable patterns, and this is our primary concern here. (Note that such a result was achieved for the positive definite completion problem [9].) Before moving on, also note that we occasionally use the notation $A([i,j,k],[i,j,k])$ to refer to the submatrix in a matrix A consisting of rows i,j, and k as well as columns i,j, and k.

Another useful tool in analyzing TP matrices is ratios between minors. We can derive necessary ratios from a special case of Sylvestor's identity for the determinant of a matrix.

Lemma 1.0.1. ([3]) Let A be an $n-by-n$ matrix partitioned as follows

$$A = \begin{bmatrix} a_{11} & a_{12}^T & a_{13} \\ a_{21} & A_{22} & a_{23} \\ a_{31} & a_{32}^T & a_{33} \end{bmatrix}$$

where A_{22} is $(n-2)-by-(n-2)$ and a_{11}, a_{33} are scalars. Define the following matrices

$$B = \begin{bmatrix} a_{11} & a_{12}^T \\ a_{21} & A_{22} \end{bmatrix}, C = \begin{bmatrix} a_{12}^T & a_{13} \\ A_{22} & a_{23} \end{bmatrix}, D = \begin{bmatrix} a_{21} & A_{22} \\ a_{31} & a_{32}^T \end{bmatrix}, E = \begin{bmatrix} A_{22} & a_{23} \\ a_{32}^T & a_{33} \end{bmatrix}$$

. Provided that $A_{22} \neq 0$, we have that

$$\det(A) = \frac{detBdetE - detCdetD}{detA_{22}}$$

Next, we introduce a new idea, upon which we focus: expansion.

Definition 1.0.1. A one-line *expansion* of an m-by-n pattern, $\mathcal{P}$, is an $m+1-by-n$ or $m-by-n+1$ pattern, $\mathcal{P}'$, in which a line of $\mathcal{P}$, with at least one unsepcified entry, has been duplicated next to the original line, so as to form $\mathcal{P}'$. This operation may be repeated, and any resulting pattern is referred to as an expansion of the original one. A sequence of such one-line expansions is also called an expansion.

Example 1.0.1. The 2-by-4 pattern below may be expanded to a $3-by-5$ pattern. The first row is duplicated to form a 3-by-4 pattern, then the new first column is duplicated to form the 3-by-5 pattern. Note that the new lines are adjacent to the lines they duplicated.

$$\begin{bmatrix} x & x & ? & x \\ ? & x & x & x \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{?} & \mathbf{x} \\ x & \mathbf{x} & x & ? & x \\ ? & \mathbf{?} & x & x & x \end{bmatrix}$$

We next describe an important criterion for completion in the single-unspecified-entry case, and a consequence. These are the single unspecified entry TP-completable patterns.

Lemma 1.0.2. ([3, Thm 2.11]) Let A be an m-by-n partial TP matrix in which $4 \leq m \leq n$ and in which the only unspecified entry lies in the (s,t) position. Any such A has a TP completion if and only if $s+t \leq 4$ or $s+t \geq m+n-2$.

From [3] we know that in the 3-by-n case, any partial TP matrix with one unspecified entry is TP-completable. The above Theorem states that if an m-by-n matrix has only one unspecified entry and that entry lies in one of the positions labeled below as “x”, then the matrix is TP-completable.

Example 1.0.2.

$$\begin{bmatrix} x & x & x & \cdots & \cdots & \cdots & \cdots \\ x & x & \cdots & \cdots & \cdots & \cdots & \cdots \\ x & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x \\ \cdots & \cdots & \cdots & \cdots & \cdots & x & x \\ \cdots & \cdots & \cdots & \cdots & x & x & x \end{bmatrix}$$

Since this remains true for 3-by-n matrices, we can directly derive [3, Theorem 2.88], that all partial 3-by-3 TP matrices with exactly one unspecified entry are TP-completable.

Now, we describe a few supporting results that will yield some existing characterizations of TP-completable matrices. The transpose of a matrix is familiar, and transposition preserves total positivity. But, there is a less familiar operation on the entries of a matrix that preserves total positivity. Define the *forward-backward reversal* of a matrix, by reading it from right to left, bottom to top, rather than in the usual way. This is simply similarity by the “backward” identity matrix, or the forward-backward reversal of the identity matrix. From [4], it follows that the transpose and the forward-backward reversal operations preserve all properties of total positivity as well as completability/non-completability of partial matrices and patterns.

Example 1.0.3. Below, the first pattern is not TP-completable by results from [3], which means its transpose, the second matrix, is also not TP-completable.

$$\begin{bmatrix} x & ? & x \\ x & x & ? \\ x & x & x \end{bmatrix} \rightarrow \begin{bmatrix} x & x & x \\ ? & x & x \\ x & ? & x \end{bmatrix}$$

Example 1.0.4. The first matrix below is TP-completable by results from [3], which means its forward-backward reversal, the second matrix, is also TP-completable.

$$\begin{bmatrix} x & x & ? \\ ? & x & x \\ x & x & x \end{bmatrix} \rightarrow \begin{bmatrix} x & x & x \\ x & x & ? \\ ? & x & x \end{bmatrix}$$

Bordering and line insertion are also helpful tools in TP completion problems. First, consider bordering.

Lemma 1.0.3. (Lemma 2.4 in [3]) Let A be an m-by-n partial TP matrix. Then there exist positive vectors x, u, v, w such that the augmented matrix $\begin{bmatrix} A|x \end{bmatrix}$, $\begin{bmatrix} u|A \end{bmatrix}$, $\begin{bmatrix} A \\ v \end{bmatrix}$, and $\begin{bmatrix} w \\ A \end{bmatrix}$ are all partial TP.

Simply put, it is possible to add a line on any side of a TP matrix so that it remains a larger TP matrix, and this extends to partial TP matrices. An image describing the idea behind these line additions can be found on the cover of [7]. For Example, if we insert a line above the matrix, then we add each value from right to left so that they are sufficiently large, as each additional entry will factor positively into any minor that it is a part of (at that point in the process). Furthermore, it was shown in [5] that line insertions between adjacent rows or columns are also possible, in such a way as to preserve total positivity.

Theorem 1.0.4. ([5, Thm 2.3]) Let A be a TP matrix. Then, a line can be inserted between any pair of adjacent lines in A so that the resulting matrix is TP.

An algorithm for the insertion can be found in [5]. By scaling the matrix row with diagonal similarity operations, we see that a line with a single specified entry can also be inserted so that the resulting matrix is TP. This is called *singly constrained line insertion*. The notion of *doubly constrained line insertion* will also be mentioned.

Example 1.0.5. The pattern $\mathcal{P}'$ is TP-completable. We simply need to insert a singly constrained line between columns 3 and 4 of $\mathcal{P}$, which we now know is always possible in a TP matrix, assuming the single constraint is in agreement with the conditions of partial total positivity.

$$\mathcal{P} = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}, \quad \mathcal{P}' = \begin{bmatrix} x & x & x & \mathbf{x} & x \\ x & x & x & \mathbf{?} & x \\ x & x & x & \mathbf{?} & x \\ x & x & x & \mathbf{?} & x \end{bmatrix}$$

Chapter 2

Northwest-southeast property

We now introduce the northwest-southeast property, which proves useful in completing matrices with clusters of unspecified entries. A partial matrix has the northwest-southeast property if for any unspecified entry, there is either no specified entry above and to the left of it or below and to the right of it. We also extend the property to include matrices with a cluster of unspecified entries in both the northwest corner and the southeast corner that fit this description.

Example 2.0.1. The following pattern has the northwest-southeast property.

$$\begin{bmatrix} ? & ? & ? & x & x \\ ? & ? & x & x & x \\ ? & ? & x & x & x \\ x & x & x & ? & ? \\ x & x & x & ? & ? \end{bmatrix}$$

Lemma 2.0.1. Any partial TP matrix with the northwest-southeast property has a TP completion.

Proof. Suppose a partial TP matrix A with the northwest-southeast property has an unspecified entry in the northwest corner. We begin with the southeastern most unspecified entry among these. It will factor in positively to the equation of any minor that we complete when choosing a value for the entry, as it will be in the (1,1) position of any new minors. Hence, we can choose this value to be large enough to keep the resulting partial matrix partial TP. We then follow the same logic, completing entries from right to left, bottom to top, until the northwest corner of A has been completed. If there is also a cluster of unspecified entries in the southeast corner then we can use the same strategy by forward-backward symmetry. If not then we are done. $\square$

Definition 2.0.1. Now, suppose a partial matrix A, does not have the northwest-southeast property, but there exists an m-by-n partial submatrix A' which has the northwest-southeast property with one cluster of unspecified entries. If the $(1,1)$ entry of A' lies in one of the six northwestern positions given by Example 1.2 (illustration of Lemma 1.0.2) in A, or the (m,n) entry of A' lies in one of the six southeastern ones, then A' has the "modified" northwest-southeast property.

Theorem 2.0.2. Any partial TP matrix with the modified northwest-southeast property has a TP completion.

Proof. Consider the following matrix

$$A = \begin{bmatrix} x & x & x & x & x & x & x \\ x & ? & ? & x & x & x & x \\ x & ? & ? & x & x & x & x \\ x & ? & x & x & x & x & x \\ x & x & x & x & x & x & x \\ x & x & x & x & x & x & x \end{bmatrix}$$

There is an unspecified entry in the $(2,2)$ position, which is a completable single entry position. Begin with the entry in the $(4,2)$ position. We need not consider the lines with unspecified entries above and to the left of this entry, as completing it will not fully complete any minors that include rows 2 or 3. So we look at the submatrix $A(\{1,4,5,6\},\{1,3,4,5,6,7\})$. In this submatrix, the entry is now a single unspecified entry in the $(2,2)$ position, so we can complete it. Using this procedure, moving through the unspecified entries from right to left and bottom to top, we can sequentially complete A. Using forward-backward reversal, we need only consider matrices with clusters of unspecified entries in the northwest corner to obtain the general case. We then move through the unspecified entries from right to left and bottom to top, using the single entry result from Lemma 1.0.2. $\square$

Completion of a pattern in this way is referred to as *sequential completion*, and it is a common technique, especially when considering expanded patterns.

Chapter 3

Expansion of TP-completable patterns

We now begin exploring the preservation of TP-completability after the expansion operation.

Theorem 3.0.1. Any expansion of a TP-completable pattern with one unspecified entry is also TP-completable.

Proof. Because expansion is a series of row/column duplications, if a pattern $\mathcal{P}$ has a single unspecified entry in a completable position, then any expansion must also have at least one unspecified entry in that same position (see Example 2.1). The result follows from Theorem 2.0.2. $\square$

Example 3.0.1. In both the original and the expanded pattern in the following pattern, there is an unspecified entry in the $(2,2)$ position.

$$\begin{bmatrix} x & x & x & x \\ x & ? & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \rightarrow \begin{bmatrix} x & x & x & x & x & x \\ x & ? & ? & ? & x & x \\ x & ? & ? & ? & x & x \\ x & ? & ? & ? & x & x \\ x & ? & ? & ? & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix}$$

We now give a proof for another main result and helpful concept in completion problems.

Theorem 3.0.2. Insertion of a line in any position of a TP matrix, with 2 entries given that are consistent with partial total positivity, may be carried out so that the result is a TP matrix.

Proof. We will begin with notation. Suppose that the m-by-n matrix $A = a_{ij}$ is TP. We will show that we can insert a doubly constrained column between columns i and $i+1$ where $i < n$. Since transposition preserves total positivity, we only need consider the case where we insert a column. We denote the result of inserting the column vector w by $A(w)$. Suppose the two constraints are in rows l and k where $1 \leq k < l \leq m$ and the remaining entries of the inserted vector are unspecified (free to be chosen). This means that we wish to complete the following partial TP matrix, $A(w)$:

$$
\left[\begin{array}{ccc|c|ccc}
x & \cdots & x & ? & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x & \cdots & x & ? & x & \cdots & x \\
x & \cdots & x & w_k & x & \cdots & x \\
x & \cdots & x & ? & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & ? & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & w_l & \vdots & \vdots & \vdots \\
x & \cdots & x & ? & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x & \cdots & x & ? & x & \cdots & x
\end{array}\right]
$$

Now, we will show that a partial TP matrix of this form is TP-completable. Since total positivity is invariant under positive scaling, we assume without loss of generality that every entry in row k is 1. Then, since the two rows k and l must be TP (the only constraints on the specified entries of the inserted column), we have $a_{l,i} < w_l < a_{l,i+1}$, as well as $w_k = 1$. Denote columns i and $i+1$ of A by C_i and C_{i+1}. Our partial TP matrix now looks like

$$
\left[\begin{array}{ccc|c|ccc}
x & \cdots & x & ? & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x & \cdots & x & ? & x & \cdots & x \\
1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
x & \cdots & x & ? & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & ? & \vdots & \vdots & \vdots \\
\vdots & \vdots & a_{l,i} & w_l & a_{l,i+1} & \vdots & \vdots \\
x & \cdots & x & ? & x & \cdots & x \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x & \cdots & x & ? & x & \cdots & x
\end{array}\right]
$$

If we inserted C_i or C_{i+1} between the columns i and $i+1$, the resulting matrix

$A(C_i)$ or $A(C_{i+1})$ would be TN. Since both the set of TN completions and the set of TP completions are convex as a consequence of the linearity of the determinant, we have that for any t, $0 < t < 1$, $A(tC_i + (1-t)C_{i+1})$ is TN. Let $x_t = tC_i + (1-t)C_{i+1}$. Now the k^{th} entry of x_t is 1 and the $l-th$ entry will be any number strictly between $a_{l,i}$ and $a_{l,i+1}$, depending upon t.

Now, let y be a vector, so that the insertion $A(y)$ is TP, according to [5]. As discussed, we may assume $y_k = 1$. Then $a_{l,i} < y_l < a_{l,i+1}$, as rows k and l of the result must form a TP matrix. If $y_l = w_l$, $A(y)$ is rhe desired insertion, and we are done. Suppose not and $y_l < w_l$ (the case where $w_l < y_l$ is similar and will be omitted). Now, choose t above so that $(x_t)_l > w_l$ and then choose s, $0 < s < 1$, so that $sy_l + (1-s)(x_t)_l = w_l$. Let $z = sy + (1-s)x_t$. We show that $A(z)$ is TP and that z is the desired doubly constrained TP line insertion that completes the proof.

Since $z = 1$ and $z_l = w_l$, z meets the two constraints. To see that $A(z)$ is TP, recall that we only need to check its initial minors [3]. A minor is initial if both its row and column indices are consecutive and at least one of the index sets begins with 1. Of course, we only need to check minors that include the column z. Since column z is positive, the only initial minors that need to be checked, must also include column C_i or C_{i+1} or both. The relevant cases are

- Case 1a: C_i and z
- Case 1b: z and C_{i+1}
- Case 2: C_i, z, and C_{i+1}

Since cases 1a and 1b are similar, we will only do 1a.

First consider an initial minor of case 2. Use C_i and C_{i+1} to eliminate that portion of z that has a positive coefficient on either C_i or C_{i+1}. This leaves a minor that is a positive scaling of the corresponding minor in $A(y)$. But this is positive, so that our minor in $A(z)$ is positive.

Now, consider an initial minor of case 1A. Use C_i to eliminate that portion of z that has a positive coefficient on C_i. This leaves our inserted column that is a positive linear combination of y and C_{i+1}. By the linearity of the determinant, this is a positive linear combination of the corresponding minor in $A(y)$ and the corresponding minor in $A(C_{i+1})$.

The former is positive and the latter is a minor of A (because column C_{i+1} is missing but a positive multiple of it appears in the inserted column). Thus, our initial minor is positive in this case as well, completing the proof. $\square$

Next, we note that while singly constrained line insertion was increased to doubly constrained line insertion, we cannot perform triply constrained line insertion. Consider a 4-by-4 TP matrix and insert a column with the first, second, and fourth entries specified between columns 2 and 3. The result is a 4-by-5 partial TP matrix in which the 3,3 entry is the only unspecified one. According to [3], there is partial TP data for this pattern with no TP completion. So the triply constrained TP line insertion is not generally possible. Of course, some triply, or more, constrained line insertions do end up yielding completable patterns depending upon the relative position of the unspecified entries. A complete analysis of this needs to occur in a broader context, but in general the result will not be TP completable.

Doubly constrained TP line insertion has some nice consequences for TP completability theory. Call a pattern *type 2* if every column is one of two types: either full (all entries specified) or a column with exactly 2 specified entries (and all of these have the same pattern). There is no restriction on the number of columns of each type or the order. A *row type 2* pattern is just the transpose of a *column type 2* matrix.

Example 3.0.2. The following pattern is type 2:

$$\begin{bmatrix} x & x & x & x & x & x \\ x & ? & x & x & ? & x \\ x & x & x & x & x & x \end{bmatrix}$$

Now, we may show that every type 2 pattern is TP completable.

Theorem 3.0.3. Any partial TP matrix that has a type 2 pattern is TP-completable.

Proof. Take a matrix A of type 2 and assume it has unspecified entries in more than one column. If there is only one column containing unspecified entries, then we are done by Theorem 3.0.2. When completing one of the columns in A, all new entries will be in the same rows in which the other partial columns have unspecified entries. Hence, when we complete a partial column in A, we do not complete any minors that contain entries in other partial columns. This means we can consider the partial matrix that contains only the column we are currently trying to complete, as well as other fully specified columns. This is again possible by the previous Theorem, and completing each partial column sequentially in this way will yield a TP completion of A. □

A *border pattern* is one where all positions are specified in the first and last rows and columns. However, all other positions are unspecified. It has been shown that any partial TP matrix having a border pattern (below) is completable. Now, we may prove this in a different way. The TP-completability of 3-by-n patterns with all unspecified entries in the same row then follows.

Example 3.0.3. A matrix with a border pattern, having specified entries on the outside of the matrix and unspecified entries in all interior spots, is type 2.

$$\begin{bmatrix} x & x & \dots & \dots & x & x \\ x & ? & \dots & \dots & ? & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x & ? & \dots & \dots & ? & x \\ x & x & \dots & \dots & x & x \end{bmatrix}$$

To further extend this idea, consider a partial TP matrix A, and suppose without loss of generality that A is column type 2, with respect to the submatrix consisting of fully specified columns and doubly constrained columns. Let the specified entries for doubly constrained columns lie in rows k_1 and k_2. Then, suppose that all other unspecified entries lie in either a fully unspecified column, or a singly constrained column with the unspecified entry in row k_1 or in row k_2. We call this pattern *type 3*, and we can describe the same criterion in terms of rows rather than columns by taking the transpose of A. The following Lemma follows from the same logic as the proof for Theorem 3.0.3.

Lemma 3.0.4. Any partial TP matrix that has a type 3 pattern is TP-completable.

We now can examine the expansion of these patterns.

Theorem 3.0.5. Any expansion of a partial TP matrix having a type 2 pattern is TP-completable.

Proof. Consider a type 2 matrix pattern $\mathcal{P}$. An expansion of $\mathcal{P}$ will simply be a larger partial TP matrix of type 2 by the definition of expansion. Hence, we can complete it to be a TP matrix. □

TP-completability of type 3 pattern expansions does not follow automatically. An expansion of a type 3 pattern could yield a larger matrix in which the original doubly constrained lines now contain more than 2 specified entries. While incomplete, we can also use the idea of Manhattan convexity as well as the northwest-southeast property to complete some expanded patterns, specifically expansions of 3-by-3 patterns.

3.1 Manhattan Convexity

Consider the following expansion of a TP-completable 3-by-3 pattern with two unspecified entries.

$$\begin{bmatrix} ? & ? & x & x & x \\ ? & ? & x & x & x \\ x & x & ? & ? & x \\ x & x & ? & ? & x \\ x & x & x & x & x \end{bmatrix}$$

A natural question arises of whether or not we can first consider the lower right cluster of unspecified entries while ignoring the upper left cluster, complete it, and then use the northwest-southeast property to complete the other cluster. We call a matrix with pieces "missing", or one that is not square or rectangular, a generalized matrix. It then becomes relevant to compare Lemma 1.0.2 to the completable single unspecified entry positions in a generalized matrix.

Definition 3.1.1. [6] A shape S is *Manhattan convex* if it is horizontally and vertically path connected and if for any nonnegative integers k_1, k_2, we have $(i, j) \in S$ whenever $(i + k_1, j), (i - k_2, j) \in S$ or $(i, j + k_1), (i, j - k_2) \in S$.

Example 3.1.1. If we take the matrix (1) and remove the northwest entries, we get the generalized matrix (2). We would like to know if the single unspecified entry in

(2) is completable, as it is in a standard matrix.

$$(1)\begin{bmatrix} ? & ? & x & x & x \\ ? & ? & x & x & x \\ x & x & x & x & x \\ x & x & x & ? & x \\ x & x & x & x & x \end{bmatrix} \to (2)\begin{bmatrix} & & x & x & x \\ & & x & x & x \\ x & x & x & x & x \\ x & x & x & ? & x \\ x & x & x & x & x \end{bmatrix} \to (3)\begin{bmatrix} D & C & x & x & x \\ B & A & x & x & x \\ x & x & x & x & x \\ x & x & x & ? & x \\ x & x & x & x & x \end{bmatrix}$$

Although there are other unspecified entries in the pattern (1), we can complete the entries in the northwest corner by using the same technique as in Lemma 2.0.1. We are left with a pattern containing one unspecified entry in a completable location, and hence (1) is TP-completable. Namely, there exists a choice for the unspecified entry such that all submatrices that are included in (2) will be positive. Hence, the six completable positions in the southeast from Lemma 1.0.2 transfer to (2) as well.

Also, note that if an unspecified entry lies in a "good position" of a submatrix that borders the missing chunk of a Manhattan convex generalized matrix, then it is in a "good position" for the generalized matrix as well. It only interacts with minors in a rectangular portion of the larger, non-rectangular generalized matrix. The completable positions that we know for a single unspecified entry in the following generalized matrix are given by 'x' below.

$$\begin{bmatrix} & & x & x & x \\ & & x & x & \cdots \\ x & x & \cdots & \cdots & x \\ x & x & \cdots & x & x \\ x & \cdots & x & x & x \end{bmatrix}$$

The logic in Example 3.1.1 yields a small general conclusion about these positions.

Theorem 3.1.1. Suppose A is a Manhattan convex, partial TP matrix with the missing portion in the northwest (southeast) corner. Then the standard single entry results of the southeast (northwest) corner (Lemma 1.0.2) still hold.

This idea aids in the proof of the following Theorem.

Theorem 3.1.2. Any expansion of a 3-by-3 TP-completable pattern with two unspecified entries, where one of them is in the $(1,1)$ or $(3,3)$ position, is also TP-completable.

Proof. In an expansion of one of these patterns, there will be a contiguous, rectangular block of unspecified entries filling the northwest or southeast corner of the pattern. A Manhattan convexity argument tells us that we can "cut out" this portion of the pattern, and use Theorem 3.1.1. The Theorem tell us that single entry results still hold in this case, so by 2.0.2, the expanded pattern is TP-completable. $\square$

Chapter 4

Implications of line insertion and a limit of expansion

Next, after mentioning singly constrained line insertion and proving the existence of doubly constrained line insertion, we move on to a few implications as well as a counter-example to the preservation of TP-completability with expansion.

Theorem 4.0.1. Any expansion of a TP-completable 3-by-n pattern with a single unspecified entry or multiple unspecified entries in the same row/column is also TP-completable.

Proof. Let A be an 3-by-n TP-completable matrix with a single unspecified entry. In expansion, we duplicate rows or columns containing unspecified entries. In an expansion of A, any column containing unspecified entries will still only have two specified entries. The result follows from doubly constrained line insertion. □

Earlier we mentioned TP-completability of border patterns, but we can now define *general border patterns*. Given the set of all 3-by-n patterns with a single unspecified entry in the second row of one of the interior columns, general border patterns are expansions of these.

Example 4.0.1. We know these to also be completable by Theorem 4.0.1. The following pattern is the general form of a general border pattern

$$\begin{bmatrix} x & \cdots & x & x & \dots & \dots & x & x & \cdots & x \\ x & \cdots & x & ? & \dots & \dots & ? & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x & \cdots & x & ? & \dots & \dots & ? & x & \cdots & x \\ x & \cdots & x & x & \dots & \dots & x & x & \cdots & x \end{bmatrix}$$

While doubly constrained line insertion applies directly to the expansion of 3-by-n patterns, we also look at the expansion of 2-by-n patterns. We first note that many expansions of 2-by-n TP-completable patterns remain completable; however, the set of expansions is quite large, due to the fact that any 2-by-n partial TP pattern is TP-completable. A proof is given below.

Lemma 4.0.2. Any 2-by-n partial TP pattern is TP-completable.

Proof. Consider a partial TP 2-by-n matrix A. Any column with an unspecified entry has at most one specified entry in it. Hence, when choosing a value for an unspecified entry in A (a singly constrained line insertion), we do not complete any minors that interact with other columns containing unspecified entries. So, we can move through the unspecified entries, considering only the submatrix consisting of our current partially specified or unspecified column and all fully specified columns. All of these single entry patterns are completable by Theorem 1.0.4, so we can sequentially complete each entry in A. □

In addition to completable 2-by-n patterns, we also characterize 2-by-n patterns which remain TP-completable after expansion. The following comes easily from Theorem 1.0.4.

Theorem 4.0.3. Any expansion of a completable partial TP 2-by-n pattern, where there is a single unspecified entry, or there are multiple unspecified entries in the same row/column, is also completable.

Now take the following 2-by-4 pattern and call the given expansion of it $\mathcal{P}$, for the purposes of our next result.

$$\begin{bmatrix} x & ? & x & x \\ x & x & ? & x \end{bmatrix} \rightarrow \begin{bmatrix} x & ? & x & x \\ x & ? & x & x \\ x & x & ? & x \\ x & x & ? & x \end{bmatrix}$$

Theorem 4.0.4. There exists a partial TP matrix A of pattern $\mathcal{P}$ so that A has no TP completion.

Proof. Consider the following partial TP matrix:

$$A = \begin{bmatrix} 1 & x & 1 & 1 \\ 1 & y & \frac{6}{5} & \frac{7}{5} \\ 1 & \frac{1}{2} & u & 2 \\ 1 & 1 & v & 5 \end{bmatrix}$$

We list the determinant inequalities given by $A([1,2,3],[1,2,3])$ and $A([1,2,3],[2,3,4])$, followed by those given by $A([2,3,4],[1,2,3])$ and $A([2,3,4],[2,3,4])$.

$$uy - ux + \frac{6}{5}x - y - \frac{1}{10} > 0 \quad , \quad \frac{12}{5}x - \frac{7}{5}ux - 2y + uy + \frac{1}{10} > 0$$

$$\frac{1}{2}v - u - vy + uy + \frac{3}{5} > 0 \quad , \quad \frac{5}{2}uy - vy - \frac{3}{10} - \frac{7}{10}u + \frac{7}{20}v > 0$$

By factoring out the x from the first two determinants and by factoring out the v

from the last two determinants, we get the following two inequalities:

$$\frac{2y - uy - \frac{1}{10}}{\frac{12}{5} - \frac{7}{5}u} < x < \frac{uy - y - \frac{1}{10}}{u - \frac{6}{5}} \quad , \quad \frac{u - uy - \frac{3}{5}}{\frac{1}{2} - y} < v < \frac{\frac{5}{2}uy - \frac{7}{10}u - \frac{3}{10}}{y - \frac{7}{20}}$$

We can focus on the following inequalities that are in terms of u, v, and y.

$$\frac{2y - uy - \frac{1}{10}}{\frac{12}{5} - \frac{7}{5}u} < \frac{uy - y - \frac{1}{10}}{u - \frac{6}{5}} \quad , \quad \frac{u - uy - \frac{3}{5}}{\frac{1}{2} - y} < \frac{\frac{5}{2}uy - \frac{7}{10}u - \frac{3}{10}}{y - \frac{7}{20}}$$

Cross multiply each inequality and move all variables to one side. In order to have an interval in which to choose the value of x, we need equation 4.1 to hold true, and to choose a value for y, we need equation **??** to hold true.

$$0 < -\frac{9}{25} + \frac{6}{25}u + \frac{3}{5}uy - \frac{2}{5}u^2 y \tag{4.1}$$

$$0 < -\frac{18}{25} + \frac{9}{5}y + \frac{6}{5}uy - 3uy^2 \tag{4.2}$$

We can factor the x and v constraints so that we have the following two inequalities, respectively.

$$\frac{1}{25}(2u - 3)(3 - 5uy) > 0 \quad , \quad \frac{3}{25}(5y - 2)(3 - 5uy) > 0$$

Assume $3 - 5uy > 0$. Then, (1) gives $2u - 3 > 0$ and (2) gives $5y - 2 > 0$. It must be true that $u > \frac{3}{2}$ and $y > \frac{2}{5}$. However, this contradicts our assumption. A similar contradiction is reached if we assume $3 - 5uy < 0$. Hence, there are no values that satisfy both (1) and (2), and there exists data for the unspecified entries in $\mathcal{P}$ so that there is no TP completion. $\square$

This shows that expansion does not, in general, preserve TP-completability.

Chapter 5

Finishing the 3-by-n problem

We begin with a few ideas that help analyze the completability of 3-by-n patterns, and then move to a case by case analysis of specific patterns. We occasionally refer to a submatrix of some matrix A, consisting of columns i, j, and k, as $A([i, j, k])$.

Theorem 5.0.1. Suppose A is a 3-by-n partial TP matrix. We can insert a column that contains at least one unspecified entry anywhere in A so that it remains partial TP.

Proof. Take a partial TP, 3-by-n matrix A. If the column to be inserted contains two unspecified entries, then it only has one specified entry. The inserted line will not interact with any minors in A. The same is true if all entries in the column are unspecified. So, consider a column with one unspecified entry, and suppose without loss of generality that the unspecified entries are in rows 1 and 2.

The only minors in A that will interact with the newly inserted column are $2-by-2$ minors with entries in the first two rows of A. Hence, this is an unconstrained line insertion in the 2-by-n matrix consisting of entries in the first two rows of A, which again is always possible. □

We now have a new form of line insertion in which we keep some entries unspecified during insertion. An important idea in identifying patterns with no completion in general follows from theorem 5.0.1. We know that a pattern does not have a TP completion if it contains a contiguous non-completable pattern within it; however, we can now look at non-contiguous subpatterns to rule out the possible completion of larger patterns.

Theorem 5.0.2. Suppose a partial TP, 3-by-n matrix A contains a subpattern that is not completable in general to a TP matrix, whether contiguous in A or not contiguous. Then A is not TP-completable.

Proof. Suppose a pattern $\mathcal{P}$ contains a non-contiguous, non-TP-completable subpattern. First, take A to be a partial TP matrix with a modified version of the pattern $\mathcal{P}$ so that some columns are deleted in order to make the non-completable subpattern a contiguous one. We can use Theorem 5.0.1 to insert partial columns until we obtain a new matrix that still has no TP completion, but now fits the original pattern $\mathcal{P}$. □

While this helps to identify when a 3-by-n matrix does not have a TP completion, we would like to fully characterize all of the 3-by-n patterns with no completion in general. We conjecture that all 3-by-n patterns that are not TP-completable and contain two fully specified columns contain a submatrix rendered not completable by table 5.1.1. We now move to a case by case analysis to highlight all we know about the 3-by-n case of TP completion. However, there are a few underlying assumptions which we now state.

1. If there are two unspecified entries in one column, the single specified entry cannot interact with any minors in the rest of a 3-by-n matrix, so we can complete this column with line insertion if the rest of the matrix has a completion. Hence, we need not consider patterns with more than one unspecified entry in any given column.

2. In the first part of the analysis, we only consider patterns with all unspecified entries in adjacent columns, then we focus on patterns with fully specified

columns separating some unspecified entries.

3. We will not yet consider patterns that are expansions of smaller patterns, so that no unspecified entries will be in adjacent columns and in the same row.

The following Lemma will also prove useful in examining the 3-by-n case.

Lemma 5.0.3. Take an m-by-n partial matrix A and suppose row k is fully specified. If the submatrix consisting of rows $1, \ldots, k$ is PTP_2, and the submatrix consisting of rows $k, \ldots, m$, then A is PTP_2. The same is true replacing row with column.

Proof. Consider the following matrix, with the fully specified row normalized to have only ones in it.

$$\begin{bmatrix} a & b \\ 1 & 1 \\ c & d \end{bmatrix}$$

If rows one and two are TP_2 then $a > b$. If rows two and three are TP_2 then $d > c$. This implies $ad > bc$ meaning that the entire matrix is TP_2. This logic transfers to the specified $2-by-2$ minors of a partial TP matrix, so we are done. □

In other words, when completing an unspecified entry in order make sure all specified $2-by-2$ minors remain positive, if there is a fully specified row or column adjacent to the entry we need only worry about the subpattern on one side of that row or column.

5.1 3-by-n patterns with two unspecified entries

The following table comes from [3] and shows the positions of pairs of unspecified entries for which there is no TP-completion in general in a 3-by-4 pattern.

Table 5.1.1.

(1, 1)	N/A			
(1, 2)	(2, 1)	(2, 3)	(3, 1)	(3, 3)
(1, 3)	(2, 2)	(2, 4)		
(1, 4)	(2, 3)	(3, 3)		
(2, 1)	(1, 2)	(3, 2)		
(2, 2)	(1, 3)	(3, 1)	(3, 3)	

Essentially, there is not a completion in general if the two unspecified entries are in different rows and adjacent columns, except for the case where one is in the $(1, 1)$ or $(3, 4)$ positions, and the case where the pair lies in the $(1, 3)$ and $(3, 2)$ positions. The latter is non-TP-completable in a matrix larger than four columns, as there will be a 3-by-4 submatrix with unspecified entries in the $(1, 2)$ and $(3, 1)$ positions. In the former, when checking contiguous minors for positivity, those unspecified entries only interact with four columns. Hence, this table also characterizes non-TP-completable patterns of size 3-by-n.

We now characterize the TP-completability of *horizontal expansions* of these patterns with two unspecified entries, where we only duplicate columns and not rows. For a 3-by-n pattern with $n \geq 5$ all TP-completable patterns have one unspecified entry in the $(1, 1)$ or $(3, n)$ positions. Suppose without loss of generality that it is the $(1, 1)$ position. In a matrix that fits an expansion of this pattern, we can choose values for the unspecified entries in row one using the bordering technique. It is then a type 2 matrix and hence has a TP completion.

Last, consider the 3-by-4 pattern with unspecified entries in the $(1, 3)$ and $(3, 2)$ positions. Suppose a 3-by-n matrix A fits a horizontal expansion of this pattern, and that the last unspecified entry in row three appears in column k (so the first unspecified entry in row one appears in column $k+1$). We can complete the submatrix $A([1, k, k+1, n])$, and A remains partial TP by Lemma 5.0.3. We are left with a partial TP matrix where two type 2 matrices are separated by two fully specified columns.

This has a TP completion as we know from [4]. Hence, all horizontal expansions in this case preserve TP-completability.

We also know from [3] that all partial TP 3-by-3 patterns with two unspecified entries are TP-completable aside from the four shown below.

$$\begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & x \end{bmatrix} \begin{bmatrix} x & ? & x \\ x & x & ? \\ x & x & x \end{bmatrix} \begin{bmatrix} x & x & x \\ ? & x & x \\ x & ? & x \end{bmatrix} \begin{bmatrix} x & x & x \\ x & x & ? \\ x & ? & x \end{bmatrix}$$

5.2 3-by-n patterns with three unspecified entries

We know from [3] that all partial TP 3-by-3 patterns with three unspecified entries are TP-completable aside from the four shown below.

$$\begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & ? \end{bmatrix} \begin{bmatrix} x & ? & x \\ x & x & ? \\ ? & x & x \end{bmatrix} \begin{bmatrix} x & x & ? \\ ? & x & x \\ x & ? & x \end{bmatrix} \begin{bmatrix} ? & x & x \\ x & x & ? \\ x & ? & x \end{bmatrix}$$

More study is needed in order to characterize the property of expansion in this case; however, the following partial TP matrix shows that expansion does not preserve TP-completability in all of these patterns.

Example 5.2.1. Consider the following partial TP matrix.

$$\begin{bmatrix} x & ? & ? & x \\ ? & x & x & ? \\ x & x & x & x \end{bmatrix} \rightarrow \begin{bmatrix} 20 & x & y & 5 \\ w & 30 & 2 & z \\ 1 & 1 & 1 & 1 \end{bmatrix} = A$$

Note that w, x, y, and z are unspecified entries, and that A fits an expansion of a 3-by-3 pattern with three unspecified entries. In order to find a TP completion, we must choose $w > 30$. However, we also need $5w < 20z$, or $w < 4z$. When choosing

z, we must maintain $z < 2$, so that $4z$ can only be as large as 8. This contradicts $w > 30$, and hence A has no TP completion.

5.2.1 The $n = 4$ case

If the three unspecified entries are all in different rows and columns (remember we assume all unspecified entries are in adjacent columns) then the 3-by-4 pattern will contain one of the non-TP-completable 3-by-3 patterns shown above. It need not be contiguous as shown by Lemma 5.0.2. Hence, we examine cases where unspecified entries alternate between two rows. We first consider when the three unspecified entries are confined to two adjacent rows, yielding the following possibilities:

$$\begin{bmatrix} ? & x & ? & x \\ x & ? & x & x \\ x & x & x & x \end{bmatrix}, \begin{bmatrix} x & ? & x & ? \\ x & x & ? & x \\ x & x & x & x \end{bmatrix}, \begin{bmatrix} x & x & x & x \\ x & ? & x & x \\ ? & x & ? & x \end{bmatrix}, \begin{bmatrix} x & x & x & x \\ x & x & ? & x \\ x & ? & x & ? \end{bmatrix}$$

$$\begin{bmatrix} x & x & x & x \\ ? & x & ? & x \\ x & ? & x & x \end{bmatrix}, \begin{bmatrix} x & x & x & x \\ x & ? & x & ? \\ x & x & ? & x \end{bmatrix}, \begin{bmatrix} x & ? & x & x \\ ? & x & ? & x \\ x & x & x & x \end{bmatrix}, \begin{bmatrix} x & x & ? & x \\ x & ? & x & ? \\ x & x & x & x \end{bmatrix}.$$

All of these are non-TP-completable by Lemma 5.0.2 and the non-TP-completable 3-by-3 patterns with two unspecified entries from [3], so we now examine cases where a fully specified row lies between the two rows that contain unspecified entries. Consider the following pattern and a matrix that fits it. Suppose that A, B, and C are values consistent with partial total positivity.

$$\begin{bmatrix} x & x & ? & x \\ x & x & x & x \\ x & ? & x & ? \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & z & 1 \\ 1 & A & B & C \\ 1 & x & 1 & y \end{bmatrix}.$$

It must be true that $B < 1 < A < C$. First, we choose z so that $\frac{B}{C} < z < \frac{B}{A}$. This is possible because $\frac{B}{C} < \frac{B}{A}$. Also, $\frac{B}{C}$ and $\frac{B}{A}$ are both less than one, so $z < 1$. Hence, we can choose z so that the matrix remains partially totally positive. Then, we have a type 2 matrix which must have a completion. Using the same logic as well as forward-backward reversal, we learn that all cases of this type, shown below, are TP-completable:

$$\begin{bmatrix} x & x & ? & x \\ x & x & x & x \\ x & ? & x & ? \end{bmatrix}, \quad \begin{bmatrix} ? & x & ? & x \\ x & x & x & x \\ x & ? & x & x \end{bmatrix}, \quad \begin{bmatrix} x & ? & x & x \\ x & x & x & x \\ ? & x & ? & x \end{bmatrix}, \quad \begin{bmatrix} x & ? & x & ? \\ x & x & x & x \\ x & x & ? & x \end{bmatrix}.$$

Next we must look at expansions of these patterns, and use

$$\mathcal{P} = \begin{bmatrix} x & x & ? & x \\ x & x & x & x \\ x & ? & x & ? \end{bmatrix} \rightarrow \begin{bmatrix} x & x & x & x & ? & \dots & ? & x & x & x \\ x & x & x & x & x & x & x & x & x & x \\ x & ? & \dots & ? & x & x & x & ? & \dots & ? \end{bmatrix} = \mathcal{P}_{exp}$$

as an Example. We also can normalize row two and column one as well as label two unspecified entries α, β and two specified entries C, D for ease of discussion:

$$\begin{bmatrix} 1 & x & x & x & \alpha & \dots & \beta & x & x & x \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & ? & \dots & ? & C & x & D & ? & \dots & ? \end{bmatrix} = A$$

Suppose entries C and D are in columns j and k, respectively. Again we use Lemma 1.0.1 to see that $\det(A([1,j,k])) = (1-\alpha)(D-C) - (C-1)(\alpha-\beta)$. First, note that α and β can always be chosen so that this determinant is positive. Next, α can be chosen so that A remains partial TP by Lemma 5.0.3. Then, we choose $\beta < \alpha$ so that $\alpha - \beta$ is small enough to negate the subtraction term in our determinant. This will yield A still partial TP, but with values chosen for α and β. Next, we use line insertion to complete the submatrix $A([1, j, j+1, \ldots, k-1, k])$ to be TP. This is possible, and Lemma 5.0.3 tells us that A remains partial TP as the only minors outside of $A([1, j, j+1, \ldots, k-1, k])$ are $2-by-2$. A is now a partial TP matrix with unspecified entries only in row three, and this has a TP completion. Hence, any expansion of $\mathcal{P}$ is TP completable. We can use the same logic on the other three patterns of this type, and see that horizontal expansion of these four patterns preserves TP-completability.

5.2.2 The $n = 5$ case

When the pattern grows to one of five columns, those three unspecified entry patterns that alternated between two adjacent rows remain non-TP-completable. However, we note that even when they alternate between rows one and three, these patterns are no longer TP-completable. See that in all the possibilities shown below, there exists a 3-by-4 submatrix whose pairs of unspecified entries appear in the earlier table, making all of these patterns non-TP-completable by Lemma 5.0.2.

$$\begin{bmatrix} x & x & ? & x & x \\ x & x & x & x & x \\ x & ? & x & ? & x \end{bmatrix}, \quad \begin{bmatrix} x & ? & x & x & x \\ x & x & x & x & x \\ ? & x & ? & x & x \end{bmatrix}, \quad \begin{bmatrix} x & x & x & ? & x \\ x & x & x & x & x \\ x & x & ? & x & ? \end{bmatrix}$$

$$\begin{bmatrix} ? & x & ? & x & x \\ x & x & x & x & x \\ x & ? & x & x & x \end{bmatrix}, \quad \begin{bmatrix} x & ? & x & ? & x \\ x & x & x & x & x \\ x & x & ? & x & x \end{bmatrix}, \quad \begin{bmatrix} x & x & ? & x & ? \\ x & x & x & x & x \\ x & x & x & ? & x \end{bmatrix}$$

5.3 3-by-n patterns with four or more unspecified entries

Suppose a 3-by-n pattern $\mathcal{P}$ has four unspecified entries in adjacent columns. Again, we need only consider "up-down" or "down-up" patterns as in the previous section, where all unspecified entries, still in adjacent columns, alternate between two rows.

Consider the following partial TP matrix, with the second row normalized to be have all of its values as one.

$$\begin{bmatrix} 5.1 & ? & 5 & ? & 4.9 & ? & 1.9 & ? \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ ? & 1 & ? & 20 & ? & 21 & ? & 22 \end{bmatrix} \rightarrow \begin{bmatrix} 5.1 & a & 5 & b & 4.9 & c & 1.9 & d \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ ? & 1 & ? & 20 & ? & 21 & ? & 22 \end{bmatrix} = A$$

Let $A([i,j,k])$ represent the submatrix consisting of all three rows and columns i, j, and k. Using Lemma 1.0.1,

$$\det(A([2,4,6])) = (a-b)(21-20) - (b-c)(20-1) = (a-b) - 19(b-c) > 0$$

$$\det(A[4,6,8])) = (b-c)(22-21) - (c-d)(21-20) = (b-c) - (c-d) > 0$$

Combining these, to be able to choose a, b, c, and d the following inequality must be possible to satisfy.

$$\frac{a-b}{19} > b-c > c-d.$$

In order to keep the top two rows partial TP, the supremum of $a-b$ over all possible values of the two unspecified entries is $5.1 - 4.9 = .2$, so we can write

$$\frac{0.2}{19} = .0105 > b-c > c-d.$$

It must be true that $b > 4.9$, so in order to have $.0105 > b-c$, c can only be as small as $4.9 - .0105 = 4.8895$. However, the value of $c - d$ will then be at least $4.8895 - 1.9 = 2.9895$, a contradiction.

This contradiction comes from two different 3-by-3 minors interacting with each other, as well as the entry in the $(1, 1)$ position restricting possible values for a. Hence, one would think that a version of this pattern with fewer columns is TP-completable.

In order to keep the top two rows partial TP as we choose values for a, b, c, and d, we need to choose a value for each of the four that is within the interval created by the values directly to the left and right. For Example, the constraint for a is $5.1 > a > 5$. If we can choose values for a, b, c, and d so that the top two rows are partial TP and the determinants of $A([2, 4, 6])$ and $A([4, 6, 8])$ are positive, then by Lemma 5.0.3 the entire matrix is partial TP, as those $2 - by - 2$ minors including entries in row three must also be positive. Then, we can complete the unspecified entries in row three using line insertion. This strategy is possible when there are three or fewer unspecified entries in one of rows one and three, as the contradiction stemmed from having a fourth, so we see that this alternating pattern is TP-completable if it has fewer than eight columns.

Next, we note that the same contradiction with the closing of intervals for unspecified entries in the top row is also applicable to the following pattern, with the second and third rows of the original one swapped:

$$\begin{bmatrix} x & ? & x & ? & x & ? & x & ? \\ ? & x & ? & x & ? & x & ? & x \\ x & x & x & x & x & x & x & x \end{bmatrix}$$

5.4 3-by-n patterns with separation

We next consider patterns with a fully specified column separating columns with unspecified entries; however, we still do not worry about any patterns that are expansions of other patterns. Hence, these patterns do not have unspecified entries in the same row and adjacent columns. In [4] it is shown that if there are two fully specified columns, we can simply consider the left and right side sub patterns along with the fully specified columns separately. If they are both TP-completable then the entire pattern is TP-completable. Hence, one way to complete patterns with one fully specified separating column is to choose a value for an unspecified entry and create a pattern with two fully specified columns.

Example 5.4.1. In the following we can choose a value for the entry in the $(1, 5)$ position of the first pattern by Lemma 5.0.3.

$$\begin{bmatrix} ? & x & x & ? & x \\ x & ? & x & x & x \\ x & x & x & x & ? \end{bmatrix} \to \begin{bmatrix} ? & x & x & x & x \\ x & ? & x & x & x \\ x & x & x & x & ? \end{bmatrix}$$

Example 5.4.2. If there is a three unspecified entry pattern on one side of the fully specified column, take the pattern below, neither the third column nor the fifth column entries can be chosen without creating a non-TP-completable subpattern. Hence, the we cannot create a two column separation and say this pattern is TP-completable in general.

$$\begin{bmatrix} x & ? & x & x & x & x \\ x & x & x & x & ? & x \\ ? & x & ? & x & x & ? \end{bmatrix}$$

To characterize when we can do this, we define a type i column as a column with an unspecified entry in row i (in the 3-by-n case we only have types 1, 2, and 3 columns). Let us begin, without loss of generality, with a fully specified separating column and the partially specified columns to the left of it. In this sequence of columns to the left, if a type 2 column appears before a type 1 or 3 column then we have a non-TP-completable 3-by-3 subpattern, and hence the larger pattern is not TP-completable to begin with. Next, we note that if the following order of columns appears to the left of our fully specified column then we cannot complete a specified entry on the right side of the fully specified column in order to separate the left and right side patterns.

$$\begin{bmatrix} x \\ x \\ ? \end{bmatrix} \begin{bmatrix} x \\ ? \\ x \end{bmatrix}, \quad \begin{bmatrix} x \\ x \\ ? \end{bmatrix} \begin{bmatrix} ? \\ x \\ x \end{bmatrix}, \quad \begin{bmatrix} x \\ ? \\ x \end{bmatrix} \begin{bmatrix} ? \\ x \\ x \end{bmatrix}, \quad \begin{bmatrix} x \\ ? \\ x \end{bmatrix} \begin{bmatrix} x \\ x \\ ? \end{bmatrix}$$

This comes from generating non-contiguous patterns that are not TP-completable in general based off table 5.1.1. Then we note that by the same logic, if we wish to complete an unspecified entry on the left side of our fully specified column in order to further separate the pattern, then we cannot have the following orders of columns on the right side of our fully specified column.

$$\begin{bmatrix} x \\ x \\ ? \end{bmatrix} \begin{bmatrix} x \\ ? \\ x \end{bmatrix}, \quad \begin{bmatrix} x \\ x \\ ? \end{bmatrix} \begin{bmatrix} ? \\ x \\ x \end{bmatrix}, \quad \begin{bmatrix} x \\ ? \\ x \end{bmatrix} \begin{bmatrix} ? \\ x \\ x \end{bmatrix}, \quad \begin{bmatrix} ? \\ x \\ x \end{bmatrix} \begin{bmatrix} x \\ ? \\ x \end{bmatrix}$$

Hence, if no sequences from the first list appear on the left or if no sequences from the second list appear on the right of the fully specified column, we can attempt to choose a value for one unspecified entry and yield a pattern with two fully specified columns separating the left and right sides. We next show that if a pattern has this described property, then we can in fact separate the the left and right subpatterns further and hence it is TP-completable.

Again without loss of generality, we look to the left of the fully specified column and analyze when we can indeed complete an unspecified entry on the left side to form a two column separation. The four possibilities of left side patterns that do not yield a non-TP-completable 3-by-3 subpattern are listed below.

$$\begin{bmatrix} x & ? & x \\ x & x & x \\ ? & x & x \end{bmatrix}, \quad \begin{bmatrix} ? & x & x \\ x & ? & x \\ x & x & x \end{bmatrix}, \quad \begin{bmatrix} x & x & x \\ x & ? & x \\ ? & x & x \end{bmatrix}, \quad \begin{bmatrix} ? & x & x \\ x & x & x \\ x & ? & x \end{bmatrix}$$

All four of these are TP-completable patterns, so for any partial TP matrices of these patterns, we can find values for both unspecified entries so that the the entire matrix is TP. Hence, we can place said value for the unspecified entry in the second column first so that we have a partial TP matrix with only one unspecified entry in the first column. This single entry pattern is TP-completable, and the rest of the matrix has remained partial TP by Lemma 5.0.3. Hence, we have the following Theorem.

Theorem 5.4.1. Suppose $\mathcal{P}$ is a 3-by-n matrix pattern with one fully specified column k, and that $\mathcal{P}$ is not an expansion of any smaller pattern. Also suppose that $\mathcal{P}$ does not contain a non-TP-completable subpattern. Assume one of the following statements is true:

1. In columns $1, \ldots, k-1$, there does not exist a column of type 3 before a column of type 1 or 2.

2. In columns $k+1, \ldots, n$, there does not exist a column of type 1 after a column of type 2 or 3.

Then $\mathcal{P}$ is a TP-completable pattern.

We conjecture that if a pattern cannot be separated in this way, then it is not a TP-completable pattern, but this remains to be shown. Finally, we give a Theorem which allows us to consider a pattern of this nature in terms of its "sections" separated by fully specified columns.

Theorem 5.4.2. Suppose $\mathcal{P}$ is a 3-by-n pattern with k sets of contiguous columns containing unspecified entries, all separated by at least one fully specified column. Let S_k represent the subpattern consisting of the k^{th} set of such contiguous columns unioned with all fully specified columns in $\mathcal{P}$. If each S_k is TP-completable then $\mathcal{P}$ is TP-completable.

Proof. Suppose a partial TP matrix A has pattern $\mathcal{P}$. If one of a subpattern defined by some set of columns S_k has a TP completion, we can choose values for all unspecified entries in S_k and the partial total positivity of the A remains true by Lemma 5.0.3. Sequential completion of this nature will yield a TP completion of A. $\square$

Chapter 6

Expansion of non-TP-Completable Patterns

While we are often concerned with completing patterns, it is natural to ask whether any expansion of a non-completable pattern is also non-completable. Hence, we prove one more main result of the paper.

Theorem 6.0.1. Suppose $\mathcal{P}$ is a non-TP-completable matrix pattern with a single unspecified entry. Then, any expansion of $\mathcal{P}$ remains not TP completable.

Proof. We will use the following matrix as an Example, and assume that the specified entries are filled in with a set of data, $\mathcal{D}$, so that A is partial TP but has no TP completion.

$$A = \begin{bmatrix} x & x & x & x \\ x & x & ? & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix}$$

Now, consider the submatrix composed of rows that do not contain the unspecified entry, shown in A below.

$$\left[\begin{array}{c} \boxed{\begin{matrix} x & x & x & x \end{matrix}} \\ \begin{matrix} x & x & ? & x \end{matrix} \\ \boxed{\begin{matrix} x & x & x & x \\ x & x & x & x \end{matrix}} \end{array}\right]$$

This submatrix is TP, so we can insert a line between its third and fourth columns so that the modified matrix remains partial TP.

$$\left[\begin{array}{c} \boxed{\begin{matrix} x & x & x & x & x \end{matrix}} \\ \begin{matrix} x & x & ? & & x \end{matrix} \\ \boxed{\begin{matrix} x & x & x & x & x \\ x & x & x & x & x \end{matrix}} \end{array}\right]$$

Next, we can place an unspecified entry in the open space, and see that the specified entries in rows one, three, and four of the inserted line are not a part of any minors that interact with the second row of A. Hence, we have a matrix of the pattern below that is still partial TP, but contains the non-completable data set $\mathcal{D}$ in the submatrix of rows 1, 2, 3, and 5.

$$\mathcal{P}' = \begin{bmatrix} x & x & x & x & x \\ x & x & ? & ? & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix}$$

This means that $\mathcal{P}'$ is also non-completable. We can continue to duplicate columns/rows by inserting lines into fully specified submatrices, and placing unspecified entries in the open spots while maintaining the partial TP property. $\square$

Conjecture 6.0.1. Any expansion of a non-TP-completable pattern is also non-TP-completable.

This is likely true; however, the same strategy cannot be used to show this as in the single unspecified entry proof unless it is possible to insert a line into a partial TP matrix like we can in a TP matrix. Consider the following Example.

Example 6.0.1. The first pattern is not TP completable, but the second pattern is. Suppose we can insert a line into a partial TP matrix so that it remains partial TP, and that A is a partial TP matrix fitting the first pattern with no TP completion. Then, we could insert a line in A to get a new partial TP matrix A' that fits the second pattern. A' would also not have a TP completion. Hence, we have a contradiction and this cannot be possible in general.

$$\begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & x \end{bmatrix} \to \begin{bmatrix} x & x & ? & x \\ ? & x & x & x \\ x & x & x & x \end{bmatrix}$$

This not a counterExample to our conjecture, as this line insertion is not an expansion, but we hope a variation of this strategy will lead to a proof of Conjecture **??**.

Chapter 7

Relationship between TP and TN completion

As an aside, it is also interesting to consider the relationship between the TN and TP completion problem, as the only concrete difference lies in strict verses non-strict inequalities.

Conjecture 7.0.1. If a pattern is TN completable then it is also TP completable.

The following Example shows that the converse is certainly not true.

Example 7.0.1.

$$\begin{bmatrix} ? & x \\ x & x \end{bmatrix} \to \begin{bmatrix} ? & 1 \\ 1 & 0 \end{bmatrix}$$

The previous pattern has partial TN data as shown that has no TN completion. Hence, the pattern is TP-completable but not TN-compeltable.

However, perhaps we can prove the conjecture by looking at the contrapositive statement. Suppose $\mathcal{P}$ is a non-TP-completable pattern and A a partial TP matrix of pattern $\mathcal{P}$ with no TP completion. Call the set of data in A with no TP completion $\mathcal{D}$.

A filled with the data $\mathcal{D}$ is partial TP so it is also partial TN. The goal is now to show either that there is no TN completion, or that a perturbation of A that remains partial TN has no TN completion. If A is not TP completable, when choosing values for the unspecified entries, either all minors will necissarily be negative numbers, or some will be negative and some will be zero.

First, suppose all minors will be negative for any chosen completion. It is possible that there exists a sequence of possible values for an unspecified entry, where the sequence converges to zero as the value of a certain minor converges to zero, and inserting a value of zero will cause the minor to be zero as well. We cannot choose zero for an unspecified entry in TP completion but we can in TN completion. However, we can perturb a specified entry that enters negatively into the minor by a small amount while keeping A partial TP. This would mean that even choosing a value of zero for the given unspecified entry would not give the minor a nonnegative value. More work is needed to complete this idea for a general non-TN-completable pattern, but we can use it in the single unspecified entry case. We conclude with the following result.

Theorem 7.0.1. Any TN-completable pattern with one unspecified entry is also TP-completable.

Proof. Suppose $\mathcal{P}$ is a non-TP-completable pattern with one unspecified entry, call it z, and A is a partial TP (also partial TN) matrix with this pattern. Let $\mathcal{D}$ be a set of partial TP data for which A has no TP completion. The set of minors in A yields a finite number of functions of z that must be greater than zero, and so we have a set of inequalities on z. If A has no TP completion, then either there is no interval for z that satisfies all the inequalities, or the intersection of all the minor inequalities places an upper bound on z that is less than or equal to zero. In the former, there also does not exist a TN completion. Values of zero are allowed when performing TN completion, so in the latter we have two cases.

Case 1: If the upper bound on z is strictly less than zero then A is a partial

TN matrix with no TN completion. Hence, $\mathcal{P}$ is also non-TN-completable.

Case 2: If the upper bound on z allows for z to be zero, then we consider one of the minors that becomes nonnegative when z takes the value of zero. Partial TP data, which is also partial TN, relies on strict inequalities and so we can perturb any entry of a TP matrix by some $\epsilon > 0$ so that the matrix remains TP. Hence, we consider one of the minors which requires z to be zero and slightly decrease one of the entries that factors into the minor positively. The data remains partial TP, but now in order for the minor to be nonnegative, z must be strictly less than zero which cannot be. Hence, $\mathcal{P}$ is also non-TN-completable.

We have shown that in either case, if there exists partial TP data for $\mathcal{P}$ with no TP completion then there exists partial TN data with no TN completion. Hence, we have proved the contrapositive of the statement. $\square$

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